16

Answers and Hints

16.4 Drug Testing

We have $\mathrm{P}[A] = 1/20$ and $\mathrm{P}[\mathrm{Pos}|A] = \mathrm{P}[\mathrm{Neg}|A^c] = 19/20$. Thus

$$\begin{aligned}\mathrm{P}[A|\mathrm{Pos}] &= \frac{\mathrm{P}[\mathrm{Pos}|A]\mathrm{P}[A]}{\mathrm{P}[\mathrm{Pos}|A]\mathrm{P}[A] + \mathrm{P}[\mathrm{Pos}|A^c]\mathrm{P}[A^c]} \\ &= \frac{\mathrm{P}[\mathrm{Pos}|A]\mathrm{P}[A]}{\mathrm{P}[\mathrm{Pos}|A]\mathrm{P}[A] + [1 - \mathrm{P}[\mathrm{Neg}|A^c]]\mathrm{P}[A^c]} \\ &= \frac{(19/20)(1/20)}{(19/20)(1/20) + (19/20)(1/20)} = 1/2.\end{aligned}$$

16.11 The Greens and the Blacks

Let A be the event "A bridge hand contains at least two aces." Let B be the event "A bridge hand contains at least one ace." Let C be the event "A bridge hand contains the ace of spades."

Then $\mathrm{P}[A|B]$ is the probability that a hand contains two aces if it contains one ace, and hence is the first probability sought. Also $\mathrm{P}[A|C]$ is the probability a hand contains two aces if it contains the ace of spades, which is the second probability sought. By Bayes' Rule,

$$\mathrm{P}[A|B] = \frac{\mathrm{P}[AB]}{\mathrm{P}[B]} = \frac{\mathrm{P}[A]}{\mathrm{P}[B]},$$

where and

$$\mathrm{P}[A|C] = \frac{\mathrm{P}[AC]}{\mathrm{P}[C]}.$$

Clearly $\mathrm{P}[C] = 0.25$, since all four hands are equally likely to get the ace of spades.

To calculate $\mathrm{P}[B]$, note that the total number of hands with no aces is the number of ways to take 13 objects from 48 (the 52 cards minus the four aces), which is $\binom{48}{13}$.

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The probability of a hand having at least one ace is then

$$\mathrm{P}[B] = \frac{\binom{52}{13} - \binom{48}{13}}{\binom{52}{13}} = 1 - \frac{39 \times 38 \times 37 \times 36}{52 \times 51 \times 50 \times 49} = 0.6962.$$

The probability of at least two aces is the probability of at least one ace minus the probability of exactly one ace. We know the former, so let's calculate the latter.

The number of hands with exactly one ace is 4 times $\binom{48}{12}$, since you can choose the ace in one of four ways, and then choose any combination of 12 cards from the 48 non-aces. But

$$\frac{4 \times \binom{48}{12}}{\binom{52}{13}} = \frac{39 \times 38 \times 37}{51 \times 50 \times 49} \approx 0.4388.$$

which is the probability of having exactly one ace. The probability of at least two aces is thus

$$\mathrm{P}[A] = .6962 - .4388 = .2574$$

(to four decimal places).

Now $\mathrm{P}[AC]$ is the probability of two aces including the ace of spades. The number of ways to get the ace of spades plus one other ace is calculated as follows: take the ace of spades out of the deck, and form hands of twelve cards. The number of ways of getting no aces from the remaining cards is $\binom{48}{12}$, so the number of hands with one other ace is $\binom{51}{12} - \binom{48}{12}$. The probability of two aces including the ace of spades is thus

$$\frac{\binom{51}{12} - \binom{48}{12}}{\binom{52}{13}} = .1402.$$

Thus $\mathrm{P}[AC] = .1402$. We now have

$$\mathrm{P}[A|C] = \frac{\mathrm{P}[AC]}{\mathrm{P}[C]} = \frac{.1402}{.25} = .5608 > \frac{\mathrm{P}[AB]}{\mathrm{P}[B]} = \frac{.2574}{.6962} = .3697.$$

16.12 Laplace's Law of Succession

Suppose there are n balls in the urn, and assume the number of white balls is uniformly distributed between 0 and n. Let A_k be the even "there are k white balls," and let B_{rm} be the event "of m balls chosen

with replacement, r are white." Then $\mathrm{P}[A_k] = 1/(n+1)$, and by Bayes' Rule we have

$$\mathrm{P}[A_k|B_{rm}] = \frac{\mathrm{P}[B_{rm}|A_k]\mathrm{P}[A_k]}{\mathrm{P}[B_{rm}]}$$

Now it is easy to check that

$$\mathrm{P}[B_{rm}|A_k] = \binom{m}{r}\left(\frac{k}{n}\right)^r\left(1-\frac{k}{n}\right)^{m-r}$$

and

$$\mathrm{P}[B_{rm}] = \sum_{k=0}^{n}\mathrm{P}[A_k]\mathrm{P}[B_{rm}|A_k]. \tag{16.1}$$

The probability of choosing a white ball on the next draw is then

$$\begin{aligned}\sum_{k=0}^{n}\left(\frac{k}{n}\right)\mathrm{P}[A_k|B_{rm}] &= \sum_{k=0}^{n}\frac{k\mathrm{P}[B_{rm}|A_k]}{n(n+1)\mathrm{P}[B_{rm}]}\\ &= \frac{1}{(n+1)\mathrm{P}[B_{rm}]}\binom{m}{r}\sum_{k=0}^{n}\left(\frac{k}{n}\right)^{r+1}\left(1-\frac{k}{n}\right)^{m-r}.\end{aligned}$$

To approximate this expression, note that if n is large, equation (16.1) is a Riemann sum representing the integral

$$\begin{aligned}\mathrm{P}[B_{rm}] &\approx \frac{1}{n+1}\binom{m}{r}\int_0^1 x^r(1-x)^{m-r}\\ &= \frac{1}{(n+1)(m+1)},\end{aligned}$$

where the integral is evaluated by integration by parts r times. Replacing m by $m+1$ and r by $r+1$ in the above expression, we see that equation (16.2) is approximately

$$\frac{1}{(n+1)\mathrm{P}[B_{rm}]}\frac{m!}{r!(m-r)!}\frac{(r+1)!(m-r)!}{(m+2)(m+1)!} = \frac{r+1}{m+2}.$$

16.13 The Brain and Kidney Problem

Let A be the event "the jar contains two brains" and let B be the event, "the mad scientist pulls out a brain." Then $\mathrm{P}[A] = \mathrm{P}[A^c] = 1/2$,

$\mathrm{P}[B|A] = 1$ and $\mathrm{P}[B|A^c] = 1/2$. Then from Bayes' Rule, the probability that the remaining blob is a brain is $\mathrm{P}[A|B]$, which is given by

$$\mathrm{P}[A|B] = \frac{\mathrm{P}[B|A]\mathrm{P}[A]}{\mathrm{P}[B|A]\mathrm{P}[A] + \mathrm{P}[B|A^c]\mathrm{P}[A^c]} = \frac{1/2}{1/2 + (1/2)(1/2)} = 2/3.$$

16.14 Sexual Harassment on the Job

If you are harassed in the first 10 years, you will quit. The probability that this does *not* happen is $p_0 = (0.95)^{10} \simeq 0.598737$, so with probability $1 - p_0$ you will be harassed exactly once. After 10 years you will not quit if harassed, and since you remain for 20 more years, the expected number of harassments is one. The conclusion is that you expect to be harassed on the job exactly once.

16.15 The Value of Eye-Witness Testimony

Let G be the event "Cab that hit JD was green," let B be the event "cab that his JD was blue," let WB be the event "witness records seeing blue cab," and finally, let WG be the event "witness records seeing green cab." We have $\mathrm{P}[G] = 85/100 = 17/20$, $\mathrm{P}[B] = 15/100 = 3/20$, $\mathrm{P}[WG|G] = \mathrm{P}[WB|B] = 4/5$, $\mathrm{P}[WB|G] = \mathrm{P}[WG|B] = 1/5$. Then Bayes' Rule yields

$$\mathrm{P}[B|WB] = \frac{\mathrm{P}[WB|B]\mathrm{P}[B]}{\mathrm{P}[WB|B]\mathrm{P}[B] + \mathrm{P}[WB|G]\mathrm{P}[G]},$$

which evaluates to $12/29$.

16.17 Bill and Harry

Let A = "Bill's statement is true" and let B = "Harry says Bill's statement is true." Also let A^c = "Bill's statement is false. Then $\mathrm{P}[A] = \mathrm{P}[B] = 1/3$, and $\mathrm{P}[A^c] = 2/3$. From Bayes' Rule,

$$\begin{aligned}\mathrm{P}[A|B] &= \frac{\mathrm{P}[B|A]\mathrm{P}[A]}{\mathrm{P}[B|A]\mathrm{P}[A] + \mathrm{P}[B|A^c]\mathrm{P}[A^c]} \\ &= \frac{(1/3)(1/3)}{(1/3)(1/3) + (2/3)(2/3)} = \frac{1}{5}\end{aligned}$$

16.18 When Weakness is Strength

(a) Clearly if c aims at anybody, it should be a. If c hits a, then his payoff is $\pi_c(bc)$. But clearly $\pi_c(bc) = \pi_c(cb)/5$, since b misses c with probability 1/5, and if he misses, it is c's turn to shoot. But

$$\pi_c(cb) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{5}\pi_c(cb),$$

since either c wins immediately (with probability 1/2), or if he misses (with probability 1/2) he gets another turn with probability 1/5. This gives $\pi_c(cb) = 5/9$, so $\pi_c(bc) = 1/9$. We conclude that if c hits a, he survives with probability 1/9. What if c shoots in the air? Clearly a will shoot at b rather than c, and b will shoot at a rather than c. Thus a and b will trade shots until one is gone, and then it is c's turn to shoot. Suppose it is a who remains. Then the probability that c survives is $\pi_c(ca)$, which is clearly 1/2 (c shoots at a and if he misses, he loses the game). If b remains, the probability c wins is $\pi_c(cb)$, which we have already calculated to be 5/9. Since both 5/9 and 1/2 are greater than 1/9, c does much better by shooting in the air.

(b) Suppose first that a goes before b, so the order is cab or acb. Since c fires in the air, a will take out b for sure, so $\pi_c(cab) = \pi_c(acb) = 1/2$, since then c has exactly one chance to shoot at a. Also, $\pi_a(cab) = \pi_a(acb) = 1/2$ and $\pi_b(cab) = \pi_b(acb) = 0$. Now suppose b goes before a. Then b gets one shot at a, and hits with probability 4/5. Thus

$$\begin{aligned}\pi_c(cba) = \pi_c(bca) &= \frac{4}{5}\pi_c(cb) + \frac{1}{5}\pi_c(ca) \\ &= \frac{4}{5} \cdot \frac{5}{9} + \frac{1}{5} \cdot \frac{1}{2} \\ &= \frac{49}{90}.\end{aligned}$$

Moreover $\pi_a(cba) = \pi_a(bca) = (1/5)(1/2) = 1/10$, and $\pi_b(cba) = \pi_b(bca) = (4/5)(4/9) = 16/45$. We then have $\pi_a = (1/2)(1/2 + 1/10) = 3/10$, $\pi_b = (1/2)(16/45) = 8/45$, and $\pi_c = (1/2)(1/2 + 49/90) = 23.5/45$. Notice that $\pi_c > \pi_a > \pi_b$.

(c) If the marksman who shoots next is randomly chosen, it is still true that a and b will shoot at each other until only one of them remains.

However now clearly c prefers to have a one-on-one against b rather than against a, so c will shoot at a if given the chance. Now

$$\pi_a(ab) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{5}\pi_a(ab),$$

so $\pi_a(ab) = 5/9$ and $\pi_b(ab) = 4/9$. Similarly reasoning gives $\pi_a(ac) = 2/3$ and $\pi_c(ac) = 1/3$. Finally,

$$\pi_b(bc) = \frac{1}{2}\left(\frac{4}{5} + \frac{1}{5}\pi_b(bc)\right) + \frac{1}{2} \cdot \frac{1}{2}\pi_b(bc),$$

from which we conclude $\pi_b(bc) = 8/13$ and $\pi_c(bc) = 5/13$. Now clearly $\pi_a[a] = \pi_a(ac) = 2/3$, $\pi_b[a] = 0$, and $\pi_c[a] = 1/3$. Similarly, it is easy to check that

$$\begin{aligned}
\pi_b[b] &= \frac{4}{5}\pi_b(bc) + \frac{1}{5}\pi_b \\
\pi_a[b] &= \frac{1}{5}\pi_a \\
\pi_c[b] &= \frac{4}{5}\pi_c(bc) + \frac{1}{5}\pi_c
\end{aligned}$$

and

$$\begin{aligned}
\pi_c[c] &= \frac{1}{2}\pi_c(bc) + \frac{1}{2}\pi_c \\
\pi_b[c] &= \frac{1}{2}\pi_b(bc) + \frac{1}{2}\pi_b \\
\pi_a[c] &= \frac{1}{2}\pi_a.
\end{aligned}$$

Moving to the final calculations, we have

$$\pi_b = \frac{1}{3}\left[0 + \frac{4}{5}\pi_b(bc) + \frac{1}{5}\pi_b + \frac{1}{2}\pi_b(bc) + \frac{1}{2}\pi_b\right].$$

We can solve this for π_b, getting $\pi_b = 24/69$. The similar equation for marksman a is

$$\pi_a = \frac{1}{3}\left[\frac{2}{3} + \frac{1}{5}\pi_a + \frac{1}{2}\pi_a\right],$$

which gives $\pi_a = 20/69$. Finally,

$$\pi_c = \frac{1}{3}\left[\frac{1}{3} + \frac{4}{5}\pi_c(bc) + \frac{1}{5}\pi_c + \frac{1}{2}\pi_c(bc) + \frac{1}{2}\pi_c\right],$$

which gives $\pi_c = 25/69$. Clearly $\pi_c > \pi_b > \pi_a$, so the meek inherit the earth.

Sources

1.2 (Rasmusen 1989); 2.3 (Morrow 1994); 2.6 (Dresher 1961); 2.10 (Dresher 1961); 2.11 (Dixit and Nalebuff 1991); 2.12 (Dixit and Nalebuff 1991); 2.14 (Morrow 1994); 3.7 (Dixit and Nalebuff 1991); 3.18 (Tirole 1988); 3.19 (Kreps 1990); 4.15 (Sinervo and Lively 1996); 4.17 (Dixit and Nalebuff 1991); 4.22 (Dresher 1961); 4.23 (Dresher 1961); 4.18 (Shachat and Walker 1997); 4.19 (Shubik 1984); 4.20 (Vorob'ev 1977); 4.21 (Vorob'ev 1977); 4.25 (Dresher 1961); 4.26 (Dresher 1961); 4.30 (Weissing and Ostrom 1991); 4.30 (Weissing and Ostrom 1991); 4.32 (Mesterton-Gibbons 1991); 4.33 (Mesterton-Gibbons 1991); 4.34 (van Damme 1991); 4.39 (Kreps 1990); 4.11 (Moulin 1986); 4.8 (Moulin 1986); 4.28 (Olcina 1997); 4.39 (Cho and Kreps 1987); 4.8 (Moulin 1986); 4.13 (Arthur 1995); 4.44 (Myerson 1991); 5.4 (Dixit and Nalebuff 1991); 7.2 (van Damme 1991, Hofbauer and Sigmund 1998); 7.2 (Bendor and Swistak 1995); 7.8 (Bishop and Cannings 1975), 7.11 (Axelrod 1984); 7.12 (Riley 1979); 8.14 (Guckenheimer and Holmes 1986); 9.5 (Weibull 1995) 9.2.2 (Binmore 1993); 9.13 (Schuster and Sigmund 1983); 9.3, 9.4, 9.6, and 9.7(Hofbauer and Sigmund 1998); 9.17 (Güth and Kliemt 1994); 10.2 (Young 1998); Chapter 11 leading quote: (Aristotle 1952); 11.3.1 and 11.3.3 (Rabin 1998); 6.3 (Tirole 1988, Rotemberg and Saloner 1991); 6.9 (Dixit and Nalebuff 1991); 6.12 (Kreps 1990); 5.15 (Selten 1975, Kreps 1990); 12.10 (Tirole 1988); 12.3 (Tirole 1988); 13.6 (Bergstrom and Lachmann 1997); 13.8 (Lachmann and Bergstrom 1998); 13.10 (Gibbons 1992, Kreps 1990); 13.11 (Rasmusen 1989); 13.11 (Rasmusen 1989); 12.8 (Meyerson 1991); 13.5 (Kirkpatrick 1982); 13.6 (Alcock 1993, Grafen 1990b); 14.7 (Kreps 1990); 6.14 (Bowles and Gintis 1993b); 6.15 (Bowles and Gintis 1993a); 6.16 (Gintis 1989); 15.1 (Osborne and Rubinstein 1990); 15.2 (Osborne and Rubinstein 1990); 15.8 (Osborne and Rubinstein 1990); 15.3 (Shaked and Sutton 1984). 8.11 and 8.12 (Hofbauer and Sigmund 1998); 16.10(vos Savant September 8, 1991); 16.16(Gott 1993, Leslie 1996); 16.17(Shubik 1954);